Math 600 Day 2: Review of advanced Calculus

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Outline

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Basic Definitions

The definition of the integral of a real-valued function $f : A \to \mathbb{R}$ defined on a rectangle $A \subset \mathbb{R}^n$ is almost identical to that of the ordinary integral when $n = 1$.

Let $[a, b]$ be a closed interval of real numbers. By a partition P of $[a, b]$ we mean a finite set of points $x_0, x_1, ..., x_n$ with $a = x_0 \leq x_1 \leq ... \leq x_n = b$.

Given a closed rectangle

$$A = [a_1, b_1] \times ... \times [a_n, b_n]$$

in $\mathbb{R}^n$, a partition of A is a collection $P = (P_1, ..., P_n)$ of partitions of the intervals $[a_1, b_1], ..., [a_n, b_n]$ which divides A into closed subrectangles S in the obvious way.

Suppose now that A is a rectangle in R^n and $f : A \rightarrow \mathbb{R}$ is a bounded real-valued function. If P is a partition of A and S is a subrectangle of P (we'll simply write $S \epsilon P$), then we define

$$m_S(f) = GLB f(x) : x \epsilon S$$

$$M_S(f) = LUB f(x) : x \epsilon S.$$

Let $vol(S)$ denote the volume of the rectangle S, and define

$L(f,P) = \Sigma_{S \epsilon P} m_S(f) vol(S) =$ lower sum of f wrt P

$U(f,P) = \Sigma_{S \epsilon P} M_S(f) vol(S) =$ upper sum of f wrt P.

Given the bounded function f on the rectangle $A \subset \mathbb{R}^n$, if $LUB_P L(f,P) = GLB_P U(f,P)$, then we say that f is Riemann integrable on A, call this common value the integral of f on A, and write it as

$$\int_A f = \int_A f(x)dx = \int_A f(x_1, ..., x_n) dx_1 ... dx_n.$$

Measure Zero and Content Zero

Definition

A subset A of $\mathbb{R}^n$ has (n-dimensional) measure zero if for every $\varepsilon > 0$ there is a covering of A by a sequence of closed rectangles $U_1, U_2, ...$ such that $\Sigma_i vol(U_i) < \varepsilon$.

Remark

Note that countable sets, such as the rational numbers, have measure zero.

Definition

A subset A of $\mathbb{R}^n$ has (n-dimensional) content zero if for every $\varepsilon > 0$ there is a *finite* covering of A by closed rectangles $U_1, U_2, ..., U_k$ such that $vol(U_1) + vol(U_2) + ... + vol(U_k) < \varepsilon$.

Remark

Note that if A has content zero, then it certainly has measure zero.

Integrable Functions

Let $A \subset \mathbb{R}^n$ and let $f : A \rightarrow \mathbb{R}$ be a bounded function. For $\delta > 0$, let

$$M(a, f, \delta) = LUB\{f(x) : x \epsilon A \ \ and \ \ |x - a| < \delta\}$$

$$m(a, f, \delta) = GLB\{f(x) : x \epsilon A \ \ and \ \ |x - a| < \delta\}.$$

Then we define the oscillation, $o(f, a)$, of f at a by

$$o(f, a) = lim_{\delta \rightarrow 0}[M(a, f, \delta) - m(a, f, \delta)].$$

This limit exists because $M(a, f, \delta) - m(a, f, \delta)$ decreases as δ decreases. The oscillation of f at a provides a measure of the extent to which f fails to be continuous at a.

Theorem

Let A be a closed rectangle in $\mathbb{R}^n$ and $f : A \to \mathbb{R}$ a bounded function. Let

$$B = \{x \epsilon A : f \ \text{ is not continuous at } \ x\}.$$

Then f is Riemann integrable on A if and only if B has measure zero.

Generalizing to Bounded Sets

If $C \subset \mathbb{R}^n$, then the **characteristic function** χ_C of C is defined by $\chi_C(x) = 1$ if x lies in C and $\chi_C(x) = 0$ if x does not lie in C.

If $C \subset \mathbb{R}^n$ is a bounded set, then $C \subset A$ for some closed rectangle A. So if $f : A \to \mathbb{R}$ is a bounded function, we define

$$\int_C f = \int_A f\chi_C,$$

provided that $f\chi_C$ is Riemann integrable. According to the homework, this product will be Riemann integrable if each factor is.

Fubini's Theorem

In freshman calculus, we learn that multiple integrals can be evaluated as iterated integrals:

$$\int_{[a,b]\times[c,d]} f(x,y)dydx = \int_{[a,b]}(\int_{[c,d]} f(x,y)dy)dx.$$

The precise statement of this result, in somewhat more general terms, is known as Fubini's Theorem.
When f is continuous, Fubini's Theorem is the straightforward multi-dimensional generalization of the above formula.

When f is merely Riemann integrable, there is a slight complication, because $f(x_0, y)$ need not be a Riemann integrable function of y. This can happen easily if the set of discontinuities of f is $x_0 \times [c, d]$, and if $f(x_0, y)$ remains discontinuous at all $y \epsilon [c, d]$.

Before we state Fubini's Theorem, we need a definition.

If $f : A \to \mathbb{R}$ is a bounded function defined on the closed rectangle A, then, whether or not f is Riemann integrable over A, the LUB of all its lower sums, and the GLB of all its upper sums, both exist. They are called the **lower** and **upper integrals** of f on A, and denoted by $L \int_A f$ and $U \int_A f$, respectively.

Theorem (Fubini's Theorem.)

Let $A \subset \mathbb{R}^n$ and $A' \subset \mathbb{R}^{n'}$ be closed rectangles, and let $f : A \times A' \to \mathbb{R}$ be Riemann integrable. For each $x \epsilon A$, define $g_x : A' \to \mathbb{R}$ by $g_x(y) = f(x, y)$. Then define

$$\mathcal{L}(x) = L\int_{A'} g_x = L\int_{A'} f(x,y)dy$$

$$\mathcal{U}(x) = U\int_{A'} g_x = U\int_{A'} f(x,y)dy.$$

Then $\mathcal{L}$ and $\mathcal{U}$ are Riemann integrable over A, and

$$\int_{A\times A'} f = \int_A \mathcal{L} = \int_A (L\int_{A'} f(x,y)dy)dx$$

$$\int_{A\times A'} f = \int_A \mathcal{U} = \int_A (U\int_{A'} f(x,y)dy)dx.$$

Proof. Let P and P' be partitions of A and A', and $P \times P'$ the corresponding partition of $A \times A'$. Then

$$L(f, P \times P') = \Sigma_{S \times S' \epsilon P \times P'} m_{S \times S'}(f) vol(S \times S')$$

$$= \Sigma_{S \epsilon P}(\Sigma_{S' \epsilon P'} m_{S \times S'}(f) vol(S')) vol(S).$$

If $x \epsilon S$, then clearly $m_{S \times S'}(f) \leq m_{S'}(g_x)$. Hence

$$\Sigma_{S' \epsilon P'} m_{S \times S'}(f) vol(S') \leq \Sigma_{S' \epsilon P'} m_{S'}(g_x) vol(S') \leq L \int_{A'} g_x = \mathcal{L}(x).$$

Therefore

$$\Sigma_{S \epsilon P}(\Sigma_{S' \epsilon P'} m_{S \times S'}(f) vol(S')) vol(S) \leq L(\mathcal{L}, P).$$

Hence

$$L(f, P \times P') \leq L(\mathcal{L}, P) \leq U(\mathcal{L}, P) \leq U(\mathcal{U}, P) \leq U(f, P \times P'),$$

where the proof of the last inequality mirrors that of the first.

Since f is integrable on $A \times A'$, we have

$$LUB \;\; L(f, P \times P') = GLB \;\; U(f, P \times P') = \int_{A \times A'} f.$$

So by a squeeze argument,

$$LUB \;\; L(\mathcal{L}, P) = GLB \;\; U(\mathcal{L}, P) = \int_A \mathcal{L} = \int_{A \times A'} f.$$

Likewise, $\int_A \mathcal{U} = \int_{A \times A'} f$, completing the proof of Fubini's Theorem. $\square$

Remark

If each g_x is Riemann integrable (as is certainly the case when $f(x,y)$ is continuous), then Fubini's Theorem says

$$\int_{A\times A'} f = \int_A \left(\int_{A'} f(x,y)dy \right) dx,$$

and likewise,

$$\int_{A\times A'} f = \int_{A'} \left(\int_{A} f(x,y)dx \right) dy.$$

Remark

One can iterate Fubini's Theorem to reduce an n-dimensional integral to an n-fold iteration of one-dimensional integrals.

Partitions of Unity

Theorem

Let A be an arbitrary subset of $\mathbb{R}^n$ and let $\mathcal{U}$ be an open cover of A. Then there is a collection Φ of C^∞ functions ϕ defined in an open set containing A, with the following properties:

1. *For each $x \epsilon A$, we have $0 \leq \phi(x) \leq 1$.*
2. *For each $x \epsilon A$, there is an open set V containing x such that all but finitely many $\phi \epsilon \Phi$ are 0 on V.*
3. *For each $x \epsilon A$, we have $\Sigma_{\phi \epsilon \Phi} \phi(x) = 1$. Note that by (2) above, this is really a finite sum in some open set containing x.*
4. *For each $\phi \epsilon \Phi$, there is an open set U in $\mathcal{U}$ such that $\phi = 0$ outside some closed set contained in U.*

A collection Φ satisfying (1) - (3) is called a C^∞ partition of unity.

If Φ also satisfies (4), then it is said to be subordinate to the cover $\mathcal{U}$.

For now we will only use continuity of the functions ϕ, but in later classes it will be important that they are of class C^∞.

Proof of Theorem.

Case 1. A is compact.
Then $A \subset U_1 \cup U_2 \cup ... \cup U_k$. Shrink the sets U_i. That is, find compact sets $D_i \subset U_i$ whose interiors cover A.

Let ψ_i be a non-negative C^∞ function which is positive on D_i and 0 outside of some closed set contained in U_i.

Then $\psi_1(x) + \psi_2(x) + ... + \psi_k(x) > 0$ for x in some open set U containing A. On this set U we can define

$$\phi_i(x) = \frac{\psi_i(x)}{(\psi_1(x) + ... + \psi_k(x))}.$$

If $f : U \to [0,1]$ is a C^∞ function which is 1 on A and 0 outside some closed set in U, then

$$\Phi = \{f\phi_1, ..., f\phi_k\}$$

is the desired partition of unity.

Case 2. $A = A_1 \cup A_2 \cup A_3 \cup ...$ where each A_i is compact and $A_i \subset int(A_{i+1})$.
For each i, let

$$\mathcal{U}_i = \{U \cap (int(A_{i+1}) - A_{i-2}) : U \epsilon \mathcal{U}\}.$$

Then $\mathcal{U}_i$ is an open cover of the compact set $B_i = A_i - int(A_{i-1})$.

By case 1, there is a partition of unity Φ_i for B_i subordinate to $\mathcal{U}_i$.

For each $x \epsilon A$, the sum $\sigma(x) = \Sigma_\phi \phi(x)$, over all ϕ in all Φ_i, is really a finite sum in some open set containing x. Then for each of these ϕ, define $\phi'(x) = \frac{\phi(x)}{\sigma(x)}$. The collection of all ϕ' is the desired partition of unity.

Case 3. A is open.
Define $A_i = \{x \epsilon A : |x| \leq i \text{ and } \mathrm{dist}(x, \partial A) \geq \frac{1}{i}\}$ and then apply case 2.

Case 4. A is arbitrary.
Let B be the union of all U in $\mathcal{U}$. By case 3, there is a partition of unity for B. This is automatically a partition of unity for A. This completes the proof of the theorem.□

Change of Variable

Consider the technique of integration by "substitution". To evaluate $\int_{x=1}^{2}(x^2-1)^3 2xdx$, we may substitute

$$y = x^2 - 1,$$

$$dy = 2xdx$$

$x = 1$ iff $y = 0$, $x = 2$ iff $y = 3$.

Then

$$\int_{x=1}^{2}(x^2-1)^3 2xdx = \int_{y=0}^{3} y^3 dy$$

$$= \frac{y^4}{4}\Big|_0^3 = \frac{81}{4}.$$

If we write $f(y) = y^3$ and $y = g(x) = x^2 - 1$, where $g : [1, 2] \to [0, 3]$, then we are using the principle that

$$\int_{x=1}^{2} f(g(x))g'(x)dx = \int_{y=g(1)}^{g(2)} f(y)dy,$$

or more generally,

$$\int_{a}^{b} (f \circ g)g' = \int_{g(a)}^{g(b)} f.$$

Proof. If $F' = f$, then $(F \circ g)' = (F' \circ g)g' = (f \circ g)g'$, by the Chain Rule. So the left side is $(F \circ g)(b) - (F \circ g)(a)$, while the right side is $F(g(b)) - F(g(a))$.

Here is the general theorem that we will prove.

Theorem (Change of Variables Theorem.)

Let $A \subset \mathbb{R}^n$ be an open set and $g : A \rightarrow \mathbb{R}^n$ a one-to-one, continuously differentiable map such that $det(g'(x)) \neq 0$ for all $x \epsilon A$. If $f : g(A) \rightarrow \mathbb{R}$ is a Riemann integrable function, then

$$\int_{g(A)} f = \int_A (f \circ g)|det(g')|.$$

Proof The proof begins with several reductions which allow us to assume that $f \equiv 1$, that A is a small open set about the point a, and that $g'(a)$ is the identity matrix. Then the argument is completed by induction on n with the use of Fubini's theorem.

Step 1. Suppose there is an open cover $\mathcal{U}$ for A such that for each $U \epsilon \mathcal{U}$ and any integrable f, we have

$$\int_{g(U)} f = \int_U (f \circ g)|det(g')|.$$

Then the theorem is true for all of A.

Proof. The collection of all $g(U)$ is an open cover of $g(A)$. Let Φ be a partition of unity subordinate to this cover. If $\phi = 0$ outside of $g(U)$, then, since g is one-to-one, we have $(\phi f) \circ g = 0$ outside of U. Hence the equation

$$\int_{g(U)} \phi f = \int_U ((\phi f) \circ g)|det(g')|$$

can be written

$$\int_{g(A)} \phi f = \int_A ((\phi f) \circ g)|det(g')|.$$

Summing over all $\phi \epsilon \Phi$ shows that

$$\int_{g(A)} f = \int_A (f \circ g)|det(g')|,$$

completing Step 1.

Step 2. It suffices to prove the theorem for $f = 1$.

Proof. If the theorem holds for $f = 1$, then it also holds for $f = constant$. Let V be a rectangle in $g(A)$ and P a partition of V. For each subrectangle S of P, let f_S be the constant function $m_S(f)$. Then

$$L(f, P) = \Sigma_{S \epsilon P} m_s(f) vol(S) = \Sigma_{S \epsilon P} \int_{int(S)} f_S$$

$$= \Sigma_{S \epsilon P} \int_{g^{-1}(int(S))} (f_S \circ g)|det(g')|$$

$$\leq \Sigma_{S \epsilon P} \int_{g^{-1}(int(S))} (f \circ g)|det(g')|$$

$$= \int_{g^{-1}(V)} (f \circ g)|det(g')|.$$

Since $\int_V f = LUB_P \; L(f, P)$, this proves that

$$\int_V f \leq \int_{g^{-1}(V)} (f \circ g)|det(g')|.$$

Likewise, letting $f_S = M_S(f)$, we get the opposite inequality, and so conclude that

$$\int_V f = \int_{g^{-1}(V)} (f \circ g)|det(g')|.$$

Then, as in Step 1, it follows that

$$\int_{g(A)} f = \int_A (f \circ g)|det(g')|.$$

Step 3. If the theorem is true for $g : A \to \mathbb{R}^n$ and for $h : B \to \mathbb{R}^n$, where $g(A) \subset B$, then it is also true for $h \circ g : A \to \mathbb{R}^n$.

Proof.

$$\int_{h \circ g(A)} f = \int_{h(g(A))} f = \int_{g(A)} (f \circ h)|det(h')|$$

$$= \int_A [(f \circ h) \circ g][|det(h')| \circ g]|det(g')|$$

$$= \int_A [f \circ (h \circ g)]|det((h \circ g)')|.$$

Step 4. The theorem is true if g is a linear transformation.

Proof. By Steps 1 and 2, it suffices to show for any open rectangle U that

$$\int_{g(U)} 1 = \int_U |det(g')|.$$

Note that for a linear transformation g , we have $g' = g$. Then this is just the fact from linear algebra that a linear transformation $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ multiplies volumes by $|det(g)|$.

Proof of the Change of Variables Theorem.

By Step 1, it is sufficient to prove the theorem in a small neighborhood of each point $a \epsilon A$.

By Step 2, it is sufficient to prove it when $f \equiv 1$.

By Steps 3 and 4, it is sufficient to prove it when $g'(a)$ is the identity matrix.

We now give the proof, which proceeds by induction on n. The proof for $n = 1$ was given at the beginning of this section. For ease of notation, we write the proof for $n = 2$.

We are given the open set $A \subset \mathbb{R}^n$ and the one-to-one, continuously differentiable map $g : A \to \mathbb{R}^n$ with $det(g'(x)) \neq 0$ for all $x \epsilon A$.

Using the reductions discussed above, given a point $a \epsilon A$, we need only find an open set U with $a \epsilon U \subset A$ such that

$$\int_{g(U)} 1 = \int_U |det(g')|,$$

and in doing so, we may assume that $g'(a)$ is the identity matrix I.

If $g : A \to \mathbb{R}^2$ is given by

$$g(x) = (g_1(x_1, x_2), g_2(x_1, x_2)),$$

then we define $h : A \to \mathbb{R}^2$ by

$$h(x) = (g_1(x_1, x_2), x_2).$$

Clearly $h'(a)$ is also the identity matrix I, so that by the Inverse Function Theorem, h is one-to-one on some neighborhood U' of a with $det(h'(x)) \neq 0$ throughout U'. So we can define $k : h(U') \to \mathbb{R}^2$ by

$$k(x_1, x_2) = (x_1, g_2(h^{-1}(x))),$$

and we'll get $g = k \circ h$. Thus we have expressed g as the composition of two maps, each of which changes fewer than n coordinates ($n = 2$ here).

By Step 3, it is sufficient to prove the theorem for h and for k, each of which (in this case) changes only one coordinate. We'll prove it here for h.

Let $a\epsilon[c_1, d_1] \times [c_2, d_2]$. By Fubini's theorem,

$$\int_{h([c_1,d_1]\times[c_2,d_2])} 1 = \int_{[c2,d2]} (\int_{h([c_1,d_1]\times\{x_2\})} 1 dx_1) dx_2.$$

Define $h|_{x_2} : [c_1, d_1] \rightarrow \mathbb{R}$ by $(h|_{x_2})(x_1) = g_1(x_1, x_2)$. Then each map $h|_{x_2}$ is one-to-one and

$$det((h|_{x_2})'(x_1) = det(h'(x_1, x_2)) \neq 0.$$

Thus, by the induction hypothesis,

$$\int_{h([c_1,d_1]\times[c_2,d_2])} 1 = \int_{[c_2,d_2]} (\int_{(h|x_2)([c_1,d_1])} 1dx_1)dx_2$$

$$= \int_{[c_2,d_2]} (\int_{[c_1,d_1]} det((h|_{x_2})')(x_1,x_2)dx_1)dx_2$$

$$= \int_{[c_2,d_2]} (\int_{[c_1,d_1]} det(h')(x_1,x_2)dx_1)dx_2$$

$$= \int_{[c_1,d_1]\times[c_2,d_2]} det(h')(x_1,x_2)dx_1dx_2$$

$$= \int_{[c_1,d_1]\times[c_2,d_2]} det(h'),$$

completing the proof of the Change of Variables Theorem.